

# ON THE ENVELOPE OF THE AXES OF A SYSTEM OF CONICS PASSING THROUGH THREE FIXED POINTS\*

BY

R. E. ALLARDICE

In a recent number of the *Annals of Mathematics*† I have shown that the envelope of the asymptotes of a system of conics passing through three fixed points consists of two three-cusped hypocycloids, touching the three straight lines that join the three fixed points in pairs. I propose now to show that the envelope of the axes of the same system of conics consists of two three-cusped hypocycloids touching three concurrent straight lines.

The foci of a conic may be regarded as four of the vertices of a complete four-side circumscribing the conic, the other two vertices being the circular points at infinity; then the straight line at infinity is one diagonal line of this four-side, and the axes are the other two diagonal lines.

The coördinates of the circular points at infinity are  $(1, -e^{\alpha i}, -e^{-\beta i})$  and  $(1, -e^{-\alpha i}, -e^{\beta i})$ ; let us denote these points for the present by  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$ .

Let the equation of the conic be

$$U \equiv \lambda_1 yz + \lambda_2 zx + \lambda_3 xy = 0,$$

the three fixed points through which the conic is to pass being the vertices of the triangle of reference; and put

$$U_1 \equiv \lambda_1 y_1 z_1 + \lambda_2 z_1 x_1 + \lambda_3 x_1 y_1; \quad U_2 \equiv \lambda_1 y_2 z_2 + \lambda_2 z_2 x_2 + \lambda_3 x_2 y_2;$$

$$U'_1 \equiv x_1 \frac{\partial U}{\partial x} + y_1 \frac{\partial U}{\partial y} + z_1 \frac{\partial U}{\partial z}; \quad U'_2 \equiv x_2 \frac{\partial U}{\partial x} + y_2 \frac{\partial U}{\partial y} + z_2 \frac{\partial U}{\partial z}.$$

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† *On some curves connected with a system of similar conics*, *Annals of Mathematics*, 2d series, vol. 3 (1902), p. 154.

The equations of the tangents from the circular points at infinity are

$$U_1'^2 = 4UU_1$$

and

$$U_2'^2 = 4UU_2;$$

and the foci are the intersections of these two pairs of straight lines.

The equation  $U_2U_1'^2 = U_1U_2'^2$  evidently represents a third pair of straight lines passing through the foci, and must therefore represent the axes.

Now the condition for similarity may be expressed in the form \*

$$\sum (\lambda_1^2 \sin^2 A - 2\lambda_2\lambda_3 \sin B \sin C) = t^2(\lambda_1 \cos A + \lambda_2 \cos B + \lambda_3 \cos C)^2,$$

where  $t$  is the tangent of the angle between the asymptotes; or,

$$\sum (\lambda_1^2 - 2\lambda_2\lambda_3 \cos A) = s^2(\lambda_1 \cos A + \lambda_2 \cos B + \lambda_3 \cos C)^2,$$

that is,

$$U_1U_2 = s^2P^2,$$

where  $s$  is the secant of the angle between the asymptotes, and

$$P = \lambda_1 \cos A + \lambda_2 \cos B + \lambda_3 \cos C.$$

Hence the equations of the axes may be expressed in the form

$$U_1U_2' = sPU_1', \quad \text{or} \quad U_2U_1' = sPU_2';$$

and

$$U_1U_2' = -sPU_1', \quad \text{or} \quad U_2U_1' = -sPU_2'.$$

Using the first of these equations, we may write the tangential coördinates of the corresponding axis in the form

$$u = U_1(\lambda_2z_2 + \lambda_3y_2) - sP(\lambda_2z_1 + \lambda_3y_1),$$

$$v = U_1(\lambda_3x_2 + \lambda_1z_2) - sP(\lambda_3x_1 + \lambda_1z_1),$$

$$w = U_1(\lambda_1y_2 + \lambda_2x_2) - sP(\lambda_1y_1 + \lambda_2x_1).$$

Noticing that

$$\lambda_1(y_1z_2 + y_2z_1) + \lambda_2(z_1x_2 + z_2x_1) + \lambda_3(x_1y_2 + x_2y_1) = -2P,$$

we have, on multiplying these equations first by  $x_1, y_1, z_1$  and adding, and then by  $x_2, y_2, z_2$ , and adding,

$$V \equiv x_1u + y_1v + z_1w = -2PU_1 - 2sPU_1,$$

$$W \equiv x_2u + y_2v + z_2w = 2U_1U_2 + 2sP^2.$$

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\* See the paper entitled, *On some curves, etc.*, referred to above.

Hence, taking account of the relation  $U_1 U_2 = s^2 P^2$ , we find,

$$P = \frac{\sqrt{W}}{\sqrt{2s(s+1)}}, \quad U_1 = -\frac{V\sqrt{s}}{\sqrt{2(s+1)W}}.$$

Now writing the coördinates in the form

$$\begin{aligned} u &= (z_2 U_1 - z_1 s P) \lambda_2 + (y_2 U_1 - y_1 s P) \lambda_3, \\ v &= (z_2 U_1 - z_1 s P) \lambda_1 + (x_2 U_1 - x_1 s P) \lambda_3, \\ w &= (y_2 U_1 - y_1 s P) \lambda_1 + (x_2 U_1 - x_1 s P) \lambda_2, \end{aligned}$$

and using the equation

$$P = \lambda_1 \cos A + \lambda_2 \cos B + \lambda_3 \cos C,$$

we have, on eliminating  $\lambda_1, \lambda_2, \lambda_3$  the relation

$$\begin{vmatrix} u & 0 & z_2 U_1 - z_1 s P & y_2 U_1 - y_1 s P \\ v & z_2 U_1 - z_1 s P & 0 & x_2 U_1 - x_1 s P \\ w & y_2 U_1 - y_1 s P & x_2 U_1 - x_1 s P & 0 \\ P & \cos A & \cos B & \cos C \end{vmatrix} = 0.$$

On substituting the values of  $P$  and  $U_1$  found above, and reducing by means of the relations

$$x_2 V + x_1 W = 2(u - v \cos C - w \cos B), \text{ etc.,}$$

we finally obtain the equation of the envelope in the form

$$\begin{vmatrix} u & 0 & u \cos B + v \cos A - w & w \cos A + u \cos C - v \\ v & u \cos B + v \cos A - w & 0 & v \cos C + w \cos B - u \\ w & w \cos A + u \cos C - v & v \cos C + w \cos B - u & 0 \\ 1/(s+1) & \cos A & \cos B & \cos C \end{vmatrix} = 0,$$

or

$$\begin{aligned} & \sum [u(v \cos C + w \cos B - u) \{u \cos(B - C) - v \cos B - w \cos C\}] \\ & - \frac{2}{s+1} (v \cos C + w \cos B - u)(w \cos A + u \cos C - v)(u \cos B + v \cos A - w) = 0. \end{aligned}$$

It may be shown that this curve has the straight line at infinity for a double tangent, the circular points at infinity being the points of contact.

It must therefore be of the fourth order and have three cusps; and hence for all values of  $s$  (except  $s = -1$ ) it is a three-cusped hypocycloid.

It may easily be shown that it always touches the perpendicular bisectors of the sides of the triangle of reference; in the special case,  $s = -1$ , the curve degenerates into the points at infinity on these three lines.

The two axes envelope the same curve only in the case of the equilateral hyperbola, for which  $s = \infty$ .

STANFORD UNIVERSITY, CALIFORNIA.

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